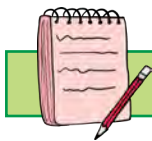


3. COMPLEX NUMBERS



Let's study.

- Complex number
- Algebra of complex number
- Solution of Quadratic equation
- Cube roots of unity



Let's recall.

- Algebra of real numbers
- Solution of linear and quadratic equations
- Representation of a real number on the number line

Introduction:

Consider the equation $x^2 + 1 = 0$. This equation has no solution in the set of real numbers because there is no real number whose square is negative. To extend the set of real numbers to a larger set, which would include such solutions.

We introduce the symbol i such that $i = \sqrt{-1}$ and $i^2 = -1$.

Symbol i is called as an **imaginary unit**.

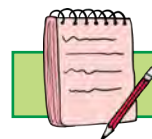
Swiss mathematician Euler (1707-1783) was the first mathematician to introduce the symbol i with $i^2 = -1$.

3.1 IMAGINARY NUMBER :

A number of the form ki , where $k \in \mathbb{R}$, $k \neq 0$ and $i = \sqrt{-1}$ is called an imaginary number.

For example

$$\sqrt{-25} = 5i, 2i, \frac{2}{7}i, -11i, \sqrt{-4} \text{ etc.}$$



Let's Note.

The number i satisfies following properties,

- $i \times 0 = 0$
- If $a \in \mathbb{R}$, then $\sqrt{-a^2} = \sqrt{i^2 a^2} = \pm ia$
- If $a, b \in \mathbb{R}$, and $ai = bi$ then $a = b$

3.2 COMPLEX NUMBER :

Definition : A number of the form $a+ib$, where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$ is called a complex number and it is denoted by z .

$$\therefore z = a+ib = a + bi$$

Here a is called the real part of z and is denoted by **Re(z) or R(z)**

' b ' is called the imaginary part of z and is denoted by **Im(z) or I(z)**

The set of complex numbers is denoted by $\mathbb{C}$

$$\therefore \mathbb{C} = \{a+ib \mid a, b \in \mathbb{R}, \text{ and } i = \sqrt{-1}\}$$

For example

z	$a+ib$	$\text{Re}(z)$	$\text{Im}(z)$
$2+4i$	$2+4i$	2	4
$5i$	$0+5i$	0	5
$3-4i$	$3-4i$	3	-4
$5+\sqrt{-16}$	$5+4i$	5	4
$2+\sqrt{-5}$	$2+\sqrt{5}i$	2	$\sqrt{5}$
$7+\sqrt{3}$	$(7+\sqrt{3})+0i$	$(7+\sqrt{3})$	0



- 1) A complex number whose real part is zero is called a imaginary number. Such a number is of the form $z = 0 + ib = ib = bi$
- 2) A complex number whose imaginary part is zero is a real number.
 $z = a + 0i = a$, for every real number.
- 3) A complex number whose both real and imaginary parts are zero is the zero complex number.
 $0 = 0 + 0i$
- 4) The set R of real numbers is a subset of the set C of complex numbers.

3.2.1 Conjugate of a Complex Number :

Definition : The conjugate of a complex number $z = a + ib$ is defined as $a - ib$ and is denoted by $\bar{z}$

For example

z	$\bar{z}$
$3 + 4i$	$3 - 4i$
$7i - 2$	$-7i - 2$
3	3
$5i$	$-5i$
$2 + \sqrt{3}$	$2 + \sqrt{3}$
$7 + \sqrt{-5}$	$7 - \sqrt{5} i$

Properties of $\bar{z}$

- 1) $(\bar{\bar{z}}) = z$
- 2) If $z = \bar{z}$, then z is real.
- 3) If $z = -\bar{z}$, then z is imaginary.

Now we define the four fundamental operations of addition, subtraction, multiplication and division of complex numbers.

3.3 ALGEBRA OF COMPLEX NUMBERS :

3.3.1 Equality of two Complex Numbers :

Definition : Two complex numbers $z_1 = a + ib$ and $z_2 = c + id$ are said to be equal if their real and imaginary parts are equal, i.e. $a = c$ and $b = d$.

For example,

- i) If $x + iy = 4 + 3i$ then $x = 4$ and $y = 3$

1) Addition :

let $z_1 = a + ib$ and $z_2 = c + id$

$$\begin{aligned} \text{then } z_1 + z_2 &= a + ib + c + id \\ &= (a + c) + (b + d)i \end{aligned}$$

Hence, $\text{Re}(z_1 + z_2) = \text{Re}(z_1) + \text{Re}(z_2)$

and $\text{Im}(z_1 + z_2) = \text{Im}(z_1) + \text{Im}(z_2)$

Ex. 1) $(2 + 3i) + (4 + 3i) = (2 + 4) + (3 + 3)i$
 $= 6 + 6i$

2) $(-2 + 5i) + (7 + 3i) + (6 - 4i)$
 $= [(-2) + 7 + 6] + [5 + 3 + (-4)]i$
 $= 11 + 4i$

Properties of addition : If z_1, z_2, z_3 are complex numbers then

- i) $z_1 + z_2 = z_2 + z_1$
- ii) $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$
- iii) $z_1 + 0 = 0 + z_1 = z_1$
- iv) $z + \bar{z} = 2\text{Re}(z)$
- v) $(\overline{z_1 + z_2}) = \bar{z}_1 + \bar{z}_2$

2) Scalar Multiplication :

If $z = a + ib$ is any complex number, then for every real number k , we define $kz = ka + i(kb)$

Ex. 1) If $z = 7 + 3i$ then
 $5z = 5(7 + 3i) = 35 + 15i$

3) Subtraction :

Let $z_1 = a + ib, z_2 = c + id$ then

$$z_1 - z_2 = z_1 + (-z_2) = (a + ib) + (-c - id)$$

$$\begin{aligned} & [\text{Here } -z_2 = -1(z_2)] \\ & = (a-c) + i(b-d) \end{aligned}$$

Hence,

$$\begin{aligned} \operatorname{Re}(z_1 - z_2) &= \operatorname{Re}(z_1) - \operatorname{Re}(z_2) \\ \operatorname{Im}(z_1 - z_2) &= \operatorname{Im}(z_1) - \operatorname{Im}(z_2) \end{aligned}$$

Ex.1) $z_1 = 4+3i, z_2 = 2+i$

$$\begin{aligned} \therefore z_1 - z_2 &= (4+3i) - (2+i) \\ &= (4-2) + (3-1)i \\ &= 2 + 2i \end{aligned}$$

Ex. 2) $z_1 = 7+i, z_2 = 4i, z_3 = -3+2i$

$$\begin{aligned} \text{then } 2z_1 - (5z_2 + 2z_3) &= 2(7+i) - [5(4i) + 2(-3+2i)] \\ &= 14 + 2i - [20i - 6 + 4i] \\ &= 14 + 2i - [-6 + 24i] \\ &= 14 + 2i + 6 - 24i \\ &= 20 - 22i \end{aligned}$$

4) Multiplication :

Let $z_1 = a+ib$ and $z_2 = c+id$. We denote multiplication of z_1 and z_2 as $z_1.z_2$ and is given by

$$\begin{aligned} z_1.z_2 &= (a+ib)(c+id) = a(c+id) + ib(c+id) \\ &= ac + adi + bci + i^2 bd \\ &= ac + (ad+bc)i - bd \quad (\because i^2 = -1) \\ z_1.z_2 &= (ac-bd) + (ad+bc)i \end{aligned}$$

Ex. $z_1 = 2+3i, z_2 = 3-2i$

$$\begin{aligned} \therefore z_1.z_2 &= (2+3i)(3-2i) = 2(3-2i) + 3i(3-2i) \\ &= 6 - 4i + 9i - 6i^2 \\ &= 6 - 4i + 9i + 6 \quad (\because i^2 = -1) \\ &= 12 + 5i \end{aligned}$$

Properties of Multiplication : If z_1, z_2, z_3 are complex numbers, then

$$\begin{aligned} \text{i)} \quad z_1.z_2 &= z_2.z_1 \\ \text{ii)} \quad (z_1.z_2).z_3 &= z_1.(z_2.z_3) \end{aligned}$$

iii) $(z_1.1) = 1.z_1 = z_1$

iv) $z.\bar{z}$ is real number.

v) $(\overline{z_1.z_2}) = \bar{z}_1 . \bar{z}_2$ (Verify)



Let's Note.

If $z = a+ib$ then $z.\bar{z} = a^2 + b^2$

We have,

$$\begin{aligned} i &= \sqrt{-1}, \quad i^2 = -1, \\ i^3 &= -i, \quad i^4 = 1 \end{aligned}$$

Powers of i :

In general,

$$\begin{aligned} i^{4n} &= 1, & i^{4n+1} &= i, \\ i^{4n+2} &= -1, & i^{4n+3} &= -i \text{ where } n \in \mathbb{N} \end{aligned}$$

5) Division :

Let $z_1 = a+ib$ and $z_2 = c+id$ be any two complex numbers such that $z_2 \neq 0$

Now,

$$\frac{z_1}{z_2} = \frac{a+ib}{c+id} \quad \text{where } z_2 \neq 0 \text{ i.e. } c+id \neq 0$$

The division can be carried out by multiplying

and dividing $\frac{z_1}{z_2}$ by conjugate of $c+id$.

SOLVED EXAMPLES

Ex. 1) If $z_1 = 3+2i$, and $z_2 = 1+i$,

then write $\frac{z_1}{z_2}$ in the form $a + ib$

Solution : $\frac{3+2i}{1+i} = \frac{3+2i}{1+i} \times \frac{1-i}{1-i}$

$$\begin{aligned}
&= \frac{3-3i+2i-2i^2}{(1)^2-(i)^2} \\
&= \frac{3-3i+2i+2}{1+1} \quad \because (i^2=-1) \\
&= \frac{5-i}{2} \\
&= \frac{5}{2} - \frac{1}{2} i
\end{aligned}$$

Ex. 2) Express $(1+2i)(-2+i)$ in the form of $a+ib$ where $a, b \in \mathbb{R}$.

Solution : $(1+2i)(-2+i) = -2+i-4i+2i^2$
 $= -2-3i-2$
 $= -4-3i$

Ex. 3) Write $(1+2i)(1+3i)(2+i)^{-1}$ in the form $a+ib$

Solution :

$$\begin{aligned}
(1+2i)(1+3i)(2+i)^{-1} &= \frac{(1+2i)(1+3i)}{2+i} \\
&= \frac{1+3i+2i+6i^2}{2+i} \\
&= \frac{-5+5i}{2+i} \times \frac{2-i}{2-i} \\
&= \frac{-10+5i+10i-5i^2}{4-i^2} \\
&= \frac{-5+15i}{4+1} \quad \because (i^2=-1) \\
&= \frac{-5+15i}{5} \\
&= -1+3i
\end{aligned}$$

Ex. 4) Express $\frac{1}{i} + \frac{2}{i^2} + \frac{3}{i^3} + \frac{5}{i^4}$ in the form of $a+ib$

Solution : We know that, $i^2=-1, i^3=-i, i^4=1$

$$\therefore \frac{1}{i} + \frac{2}{i^2} + \frac{3}{i^3} + \frac{5}{i^4}$$

$$\begin{aligned}
&= \frac{1}{i} + \frac{2}{-1} + \frac{3}{-i} + \frac{5}{1} \\
&= \frac{1}{i} - \frac{3}{i} - 2 + 5 \\
&= \frac{-2}{i} + 3 = 2i + 3 \text{ (verify!)}
\end{aligned}$$

Ex. 5) If a and b are real and $(i^4+3i)a + (i-1)b + 5i^3 = 0$, find a and b .

Solution : $(i^4+3i)a + (i-1)b + 5i^3 = 0+0i$

i.e. $(1+3i)a + (i-1)b - 5i = 0+0i$

$\therefore a + 3ai + bi - b - 5i = 0+0i$

i.e. $(a-b) + (3a+b-5)i = 0+0i$

Equating real and imaginary parts, we get

$a-b = 0$ and $3a+b-5 = 0$

$\therefore a=b$ and $3a+b = 5$

$\therefore 3a+a = 5$

i.e. $4a = 5$

or $a = \frac{5}{4}$

$\therefore a = b = \frac{5}{4}$

Ex. 6) If $x + 2i + 15i^6y = 7x + i^3(y+4)$ find $x+y$, given that $x, y \in \mathbb{R}$.

Solution :

$x + 2i + 15i^6y = 7x + i^3(y+4)$

$\therefore x + 2i - 15y = 7x - (y+4)i$

$\therefore x - 15y + 2i = 7x - (y+4)i$

Equating real and imaginary parts, we get

$x - 15y = 7x$ and $2 = -(y+4)$

$\therefore -6x - 15y = 0$ (i) $y+6 = 0$ (ii)

$\therefore y = -6, x = 15$

$\therefore x + y = 15 - 6 = 9$

Ex. 7) Find the value of $x^3 - x^2 + 2x + 4$
when $x = 1 + \sqrt{3} i$.

Solution : Since $x = 1 + \sqrt{3} i$

$$\therefore (x-1) = \sqrt{3} i$$

squaring both sides, we get

$$(x-1)^2 = (\sqrt{3} i)^2$$

$$\therefore x^2 - 2x + 1 = 3i^2$$

$$\text{i.e. } x^2 - 2x + 1 = -3$$

$$\therefore x^2 - 2x + 4 = 0$$

Now, consider

$$x^3 - x^2 + 2x + 4 = x(x^2 - x + 2) + 4$$

$$= x(x^2 - 2x + 4 + x - 2) + 4$$

$$= x(0 + x - 2) + 4$$

$$= x^2 - 2x + 4$$

$$= 0$$

Ex. 8) Show that $\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^3 = i$.

Solution :

$$\begin{aligned} \text{L.H.S.} &= \left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^3 = \left(\frac{\sqrt{3}+i}{2}\right)^3 \\ &= \frac{(\sqrt{3})^3 + 3(\sqrt{3})^2 i + 3(\sqrt{3}) i^2 + (i)^3}{(2)^3} \\ &= \frac{3\sqrt{3} + 9i - 3\sqrt{3} - i}{8} \\ &= \frac{8i}{8} \\ &= i \\ &= \text{R.H.S.} \end{aligned}$$

Ex. 9) If $x = -5 + 2\sqrt{-4}$, find the value of $x^4 + 9x^3 + 35x^2 - x + 64$.

Solution : $x = -5 + 2\sqrt{-4}$

$$\therefore x = -5 + 4i$$

$$\therefore x + 5 = 4i$$

On squaring both sides

$$(x+5)^2 = (4i)^2$$

$$\therefore x^2 + 10x + 25 = -16$$

$$\therefore x^2 + 10x + 41 = 0$$

$$\begin{array}{r} x^2 - x + 4 \\ x^2 + 10x + 41 \overline{) x^4 + 9x^3 + 35x^2 - x + 64} \\ \underline{x^4 + 10x^3 + 41x^2} \\ -x^3 - 6x^2 - x + 64 \\ \underline{-x^3 - 10x^2 - 41x} \\ 4x^2 + 40x + 64 \\ \underline{4x^2 + 40x + 164} \\ -100 \end{array}$$

$$\therefore x^4 + 9x^3 + 35x^2 - x + 64$$

$$= (x^2 + 10x + 41)(x^2 - x + 4) - 100$$

$$= 0 \times (x^2 - x + 4) - 100$$

$$= -100$$

EXERCISE 3.1

1) Write the conjugates of the following complex numbers

i) $3+i$ ii) $3-i$ iii) $-\sqrt{5} - \sqrt{7} i$

iv) $-\sqrt{-5}$ v) $5i$ vi) $\sqrt{5} - i$

vii) $\sqrt{2} + \sqrt{3} i$

2) Express the following in the form of $a+ib$, $a, b \in \mathbb{R}$ $i = \sqrt{-1}$. State the value of a and b .

i) $(1+2i)(-2+i)$ ii) $\frac{i(4+3i)}{(1-i)}$

iii) $\frac{(2+i)}{(3-i)(1+2i)}$ iv) $\frac{3+2i}{2-5i} + \frac{3-2i}{2+5i}$

v) $\frac{2+\sqrt{-3}}{4+\sqrt{-3}}$ vi) $(2+3i)(2-3i)$

$$\text{vii) } \frac{4i^8 - 3i^9 + 3}{3i^{11} - 4i^{10} - 2}$$

3) Show that $(-1 + \sqrt{3}i)^3$ is a real number.

4) Evaluate the following :

$$\begin{array}{llll} \text{i) } i^{35} & \text{ii) } i^{888} & \text{iii) } i^{93} & \text{iv) } i^{116} \\ \text{v) } i^{403} & \text{vi) } \frac{1}{i^{58}} & \text{vii) } i^{30} + i^{40} + i^{50} + i^{60} \end{array}$$

5) Show that $1 + i^{10} + i^{20} + i^{30}$ is a real number.

6) Find the value of

$$\text{i) } i^{49} + i^{68} + i^{89} + i^{110}$$

$$\text{ii) } i + i^2 + i^3 + i^4$$

7) Find the value of $1 + i^2 + i^4 + i^6 + i^8 + \dots + i^{20}$

8) Find the value of x and y which satisfy the following equations ($x, y \in \mathbb{R}$)

$$\text{i) } (x+2y) + (2x-3y)i + 4i = 5$$

$$\text{ii) } \frac{x+1}{1+i} + \frac{y-1}{1-i} = i$$

9) Find the value of

$$\text{i) } x^3 - x^2 + x + 46, \text{ if } x = 2+3i.$$

$$\text{ii) } 2x^3 - 11x^2 + 44x + 27, \text{ if } x = \frac{25}{3-4i}.$$

3.4 Square root of a complex number :

Consider $z = x+iy$ be any complex number

$$\text{Let } \sqrt{x+iy} = a+ib, a, b \in \mathbb{R}$$

On squaring both the sides, we get

$$x+iy = (a+ib)^2$$

$$x+iy = (a^2-b^2) + (2ab)i$$

Equating real and imaginary parts, we get

$$x = (a^2-b^2) \text{ and } y = 2ab$$

Solving the equations simultaneously, we can get the values of a and b .

SOLVED EXAMPLES

1) Find the square root of $6+8i$

Let the square root of $6+8i$ be $a+ib$,

$$(a, b \in \mathbb{R})$$

$$\therefore \sqrt{6+8i} = a+ib, a, b \in \mathbb{R}$$

On squaring both the sides, we get

$$6+8i = (a+ib)^2$$

$$\therefore 6+8i = a^2-b^2+2abi$$

Equating real and imaginary parts, we have

$$6 = a^2-b^2 \quad \dots (1)$$

$$8 = 2ab \quad \dots (2)$$

$$\therefore a = \frac{4}{b}$$

$$\therefore 6 = \left(\frac{4}{b}\right)^2 - b^2$$

$$\text{i.e. } 6 = \frac{16}{b^2} - b^2$$

$$\therefore b^4+6b^2-16 = 0$$

$$\text{put } b^2 = m$$

$$\therefore m^2+6m-16 = 0$$

$$\therefore (m+8)(m-2) = 0$$

$$\therefore m = -8 \text{ or } m = 2$$

$$\text{i.e. } b^2 = -8 \text{ or } b^2 = 2$$

$$\text{but } b \in \mathbb{R} \quad \therefore b^2 \neq -8$$

$$\therefore b^2 = 2 \quad \therefore b = \pm \sqrt{2}$$

$$\text{when } b = \sqrt{2}, a = 2\sqrt{2}$$

$$\therefore \text{Square root of}$$

$$6+8i = 2\sqrt{2} + \sqrt{2}i = \sqrt{2}(2+i)$$

$$\text{when } b = -\sqrt{2}, a = -2\sqrt{2}$$

$$\therefore \text{Square root of}$$

$$6+8i = -2\sqrt{2} - \sqrt{2}i = -\sqrt{2}(2+i)$$

$$\therefore \sqrt{6+8i} = \pm\sqrt{2}(2+i)$$

Ex. 2 : Find the square root of $2i$

Solution :

$$\text{Let } \sqrt{2i} = a+ib \quad a, b \in \mathbb{R}$$

On squaring both the sides, we have

$$2i = (a+ib)^2$$

$$\therefore 0+2i = a^2-b^2+2iab$$

Equating real and imaginary parts, we have

$$a^2-b^2 = 0, \quad 2ab = 2, \quad ab = 1$$

$$\text{As } (a^2+b^2)^2 = (a^2-b^2)^2 + (2ab)^2$$

$$(a^2+b^2)^2 = 0^2 + 2^2$$

$$(a^2+b^2)^2 = 2^2$$

$$\therefore a^2+b^2 = 2$$

Solving $a^2+b^2 = 2$ and $a^2-b^2 = 0$ we get

$$2a^2 = 2$$

$$a^2 = 1$$

$$a = \pm 1$$

$$\text{But } b = \frac{2}{2a} = \frac{1}{a} = \frac{1}{\pm 1} = \pm 1$$

$$\therefore \sqrt{2i} = 1+i \text{ or } -1-i$$

$$\text{i.e. } \sqrt{2i} = \pm (1+i)$$

3.5 Solution of a Quadratic Equation in complex number system :

Let the given equation be $ax^2 + bx + c = 0$ where $a, b, c \in \mathbb{R}$ and $a \neq 0$

$\therefore$ the solution of this quadratic equation is given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Hence, the roots of the equation $ax^2+bx+c = 0$

$$\text{are } \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

The expression $(b^2-4ac) = D$ is called the discriminant.

If $D < 0$ then the roots of the given quadratic equation are not real in nature, that is the roots of such equation are complex numbers.



Let's Note.

If $p + iq$ is a root of equation $ax^2 + bx + c = 0$ where $a, b, c \in \mathbb{R}$ and $a \neq 0$ then $p - iq$ is also a root of the given equation. That is complex roots occurs in conjugate pairs.

SOLVED EXAMPLES

Ex. 1 : Solve $x^2 + x + 1 = 0$

Solution : Given equation is $x^2 + x + 1 = 0$

where $a = 1, b = 1, c = 1$

So, the given equation has complex roots.

These roots are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-1 \pm \sqrt{-3}}{2}$$

$$= \frac{-1 \pm \sqrt{3}i}{2}$$

$$\therefore \text{Roots are } \frac{-1 + \sqrt{3}i}{2} \text{ and } \frac{-1 - \sqrt{3}i}{2}$$

Ex. 2 : Solve the following quadratic equation $x^2 - 4x + 13 = 0$

Solution : The given quadratic equation is

$$x^2 - 4x + 13 = 0$$

$$x^2 - 4x + 4 + 9 = 0$$

$$\therefore (x - 2)^2 + 3^2 = 0$$

$$(x - 2)^2 = -3^2$$

$$(x - 2)^2 = +3^2 \cdot i^2$$

taking square root we get,

$$(x - 2) = \pm 3i$$

$$\therefore x = 2 \pm 3i$$

$$\therefore x = 2 + 3i \text{ or } x = 2 - 3i$$

$$\text{Solution set} = \{2 + 3i, 2 - 3i\}$$

Ex. 3 : Solve $x^2 + 4ix - 5 = 0$; where $i = \sqrt{-1}$

Solution : Given quadratic equation is

$$x^2 + 4ix - 5 = 0$$

Comparing with $ax^2 + bx + c = 0$

$$a = 1, \quad b = 4i, \quad c = -5$$

Consider $b^2 - 4ac = (4i)^2 - 4(1)(-5)$

$$= 16i^2 + 20$$

$$= -16 + 20 \quad (\because i^2 = -1)$$

$$= 4$$

The roots of quadratic equation are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-4i \pm \sqrt{4}}{2(1)}$$

$$= \frac{-4i \pm 2}{2}$$

$$x = -2i \pm 1$$

Solution set = $\{-2i + 1, -2i - 1\}$

EXERCISE 3.2

1) Find the square root of the following complex numbers

i) $-8-6i$ ii) $7+24i$ iii) $1+4\sqrt{3}i$

iv) $3+2\sqrt{10}i$ v) $2(1-\sqrt{3}i)$

2) Solve the following quadratic equations.

i) $8x^2 + 2x + 1 = 0$

ii) $2x^2 - \sqrt{3}x + 1 = 0$

iii) $3x^2 - 7x + 5 = 0$

iv) $x^2 - 4x + 13 = 0$

3) Solve the following quadratic equations.

i) $x^2 + 3ix + 10 = 0$

ii) $2x^2 + 3ix + 2 = 0$

iii) $x^2 + 4ix - 4 = 0$

iv) $ix^2 - 4x - 4i = 0$

4) Solve the following quadratic equations.

i) $x^2 - (2+i)x - (1-7i) = 0$

ii) $x^2 - (3\sqrt{2} + 2i)x + 6\sqrt{2}i = 0$

iii) $x^2 - (5-i)x + (18+i) = 0$

iv) $(2+i)x^2 - (5-i)x + 2(1-i) = 0$

3.6 Cube roots of unity :

Number 1 is often called unity. Let x be a cube root of unity.

$$\therefore x^3 = 1$$

$$\therefore x^3 - 1 = 0$$

$$\therefore (x-1)(x^2 + x + 1) = 0$$

$$\therefore x-1 = 0 \text{ or } x^2 + x + 1 = 0$$

$$\therefore x = 1 \text{ or } x = \frac{-1 \pm \sqrt{(1)^2 - 4 \times 1 \times 1}}{2 \times 1}$$

$$\therefore x = 1 \text{ or } x = \frac{-1 \pm \sqrt{-3}}{2}$$

$$\therefore x = 1 \text{ or } x = \frac{-1 \pm i\sqrt{3}}{2}$$

$$\therefore \text{Cube roots of unity are, } 1, \frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}$$

Among the three cube roots of unity, one is real and other two roots are complex conjugates of each other.

Now consider

$$\left(\frac{-1+i\sqrt{3}}{2} \right)^2$$

$$= \frac{1}{4} \left[(-1)^2 + 2 \times (-1) \times i\sqrt{3} + (i\sqrt{3})^2 \right]$$

$$= \frac{1}{4} (1 - 2i\sqrt{3} - 3)$$

$$= \frac{1}{4} (-2 - 2i\sqrt{3})$$

$$= \frac{-1-i\sqrt{3}}{2}$$

Similarly it can be verified that $\left(\frac{-1-i\sqrt{3}}{2} \right)^2$

$$\frac{-1+i\sqrt{3}}{2}$$

Thus complex roots of unity are squares of each other. Thus cube roots of unity are given by

$$1, \frac{-1+i\sqrt{3}}{2}, \left(\frac{-1-i\sqrt{3}}{2}\right)^2$$

$$\text{Let } \frac{-1+i\sqrt{3}}{2} = w, \text{ then } \frac{-1-i\sqrt{3}}{2} = w^2$$

Hence, cube roots of unity are 1, w , w^2 or 1, w , $\bar{w}$

$$\text{where } w = \frac{-1+i\sqrt{3}}{2} \text{ and } w^2 = \frac{-1-i\sqrt{3}}{2}$$

w is complex cube root of 1.

$$\therefore w^3 = 1 \therefore w^3 - 1 = 0$$

$$\text{i.e. } (w-1)(w^2+w+1) = 0$$

$$\therefore w=1 \text{ or } w^2+w+1 = 0$$

but $w \neq 1$

$$\therefore w^2+w+1 = 0$$

Properties of 1, w , w^2

$$\text{i) } w^2 = \frac{1}{w} \text{ and } \frac{1}{w^2} = w$$

$$\text{ii) } w^3 = 1 \text{ so } w^{3n} = 1$$

$$\text{iii) } w^4 = w^3 \cdot w = w \text{ so } w^{3n+1} = w$$

$$\text{iv) } w^5 = w^2 \cdot w^3 = w^2 \cdot 1 = w^2$$

$$\text{So } w^{3n+2} = w^2$$

$$\text{v) } \bar{w} = w^2$$

$$\text{vi) } (\bar{w})^2 = w$$

SOLVED EXAMPLES

Ex. 1 : If w is a complex cube root of unity, then prove that

$$\text{i) } \frac{1}{w} + \frac{1}{w^2} = -1$$

$$\text{ii) } (1+w^2)^3 = -1$$

$$\text{iii) } (1-w+w^2)^3 = -8$$

Solution : Given, w is a complex cube root of unity.

$$\therefore w^3 = 1 \text{ Also } w^2+w+1 = 0$$

$$\therefore w^2+1 = -w \text{ and } w+1 = -w^2$$

$$\text{i) } \frac{1}{w} + \frac{1}{w^2} = \frac{w+1}{w^2} = \frac{-w^2}{w^2} = -1$$

$$\text{ii) } (1+w^2)^3 = (-w)^3 = -w^3 = -1$$

$$\begin{aligned} \text{iii) } (1-w+w^2)^3 &= (1+w^2-w)^3 \\ &= (-w-w)^3 \quad (\because 1+w^2 = -w) \\ &= (-2w)^3 \\ &= -8w^3 \\ &= -8 \times 1 \\ &= -8 \end{aligned}$$

Ex. 2 : If w is a complex cube root of unity, then show that

$$(1-w)(1-w^2)(1-w^4)(1-w^5) = 9$$

Solution : $(1-w)(1-w^2)(1-w^4)(1-w^5)$

$$\begin{aligned} &= (1-w)(1-w^2)(1-w^3 \cdot w)(1-w^3 \cdot w^2) \\ &= (1-w)(1-w^2)(1-w)(1-w^2) \\ &= (1-w)^2(1-w^2)^2 \\ &= [(1-w)(1-w^2)]^2 \\ &= (1-w^2-w+w^3)^2 \\ &= [1-(w^2+w)+1]^2 \\ &= [1-(-1)+1]^2 \\ &= (1+1+1)^2 \\ &= (3)^2 \\ &= 9 \end{aligned}$$

Ex. 3 : Prove that

$$1+w^n+w^{2n}=3, \text{ if } n \text{ is a multiple of } 3$$

$$1+w^n+w^{2n}=0, \text{ if } n \text{ is not multiple of } 3, n \in \mathbb{N}$$

Solution : If n is a multiple of 3 then $n=3k$ and if n is not a multiple of 3 then $n = 3k+1$ or $n = 3k+2$, where $k \in \mathbb{N}$

Case 1: If n is multiple of 3

$$\begin{aligned}\text{then } 1+w^n+w^{2n} &= 1+w^{3k}+w^{2 \times 3k} \\ &= 1+(w^3)^k+(w^3)^{2k} \\ &= 1+(1)^k+(1)^{2k} \\ &= 1+1+1 \\ &= 3\end{aligned}$$

Case 2: If $n = 3k + 1$

$$\begin{aligned}\text{then } 1+w^n+w^{2n} &= 1+(w)^{3k+1}+(w^2)^{3k+1} \\ &= 1+(w^3)^k \cdot w + (w^3)^{2k} \cdot w^2 \\ &= 1+(1)^k \cdot w + (1)^{2k} \cdot w^2 \\ &= 1+w+w^2 \\ &= 0\end{aligned}$$

Similarly by putting $n = 3k+2$, we have,

$$1+w^n+w^{2n}=0. \text{ Hence the results.}$$

EXERCISE 3.3

- 1) If w is a complex cube root of unity, show that
 - i) $(2-w)(2-w^2)=7$
 - ii) $(2+w+w^2)^3-(1-3w+w^2)^3=65$
 - iii) $\frac{(a+bw+cw^2)}{c+aw+bw^2} = w^2$
- 2) If w is a complex cube root of unity, find the value of
 - i) $w + \frac{1}{w}$ ii) $w^2+w^3+w^4$ iii) $(1+w^2)^3$
 - iv) $(1-w-w^2)^3 + (1-w+w^2)^3$
 - v) $(1+w)(1+w^2)(1+w^4)(1+w^8)$
- 3) If α and β are the complex cube roots of unity, show that

$$\alpha^2+\beta^2+\alpha\beta = 0$$

- 4) If $x=a+b$, $y=\alpha a+\beta b$, and $z=a\beta+b\alpha$ where α and β are the complex cube-roots of unity, show that $xyz=a^3+b^3$
- 5) If w is a complex cube-root of unity, then prove the following
 - i) $(w^2+w-1)^3=-8$
 - ii) $(a+b)+(aw+bw^2)+(aw^2+bw)=0$



Let's remember!

- * A number of the form $a+ib$, where a and b are real numbers, $i = \sqrt{-1}$, is called a complex number.
- * Let $z_1 = a+ib$ and $z_2 = c+id$. Then

$$z_1 + z_2 = (a+c) + (b+d)i$$

$$z_1 z_2 = (ac-bd) + (ad+bc)i$$
- * For any positive integer k ,

$$i^{4k} = 1, i^{4k+1} = i, i^{4k+2} = -1, i^{4k+3} = -i$$
- * The conjugate of complex number $z = a+ib$ denoted by $\bar{z}$, is given by $\bar{z} = a-ib$
- * The cube roots of unity are denoted by 1, w , w^2 or 1, w , $\bar{w}$

MISCELLANEOUS EXERCISE - 3

- 1) Find the value of $\frac{i^{592}+i^{590}+i^{588}+i^{586}+i^{584}}{i^{582}+i^{580}+i^{578}+i^{576}+i^{574}}$
- 2) Find the value of $\sqrt{-3} \times \sqrt{-6}$
- 3) Simplify the following and express in the form $a+ib$.
 - i) $3 + \sqrt{-64}$ ii) $(2i^3)^2$ iii) $(2+3i)(1-4i)$
 - iv) $\frac{5}{2} i(-4-3i)$ v) $(1+3i)^2(3+i)$ vi) $\frac{4+3i}{1-i}$
 - vii) $\left(1+\frac{2}{i}\right)\left(3+\frac{4}{i}\right)(5+i)^{-1}$ viii) $\frac{\sqrt{5}+\sqrt{3}i}{\sqrt{5}-\sqrt{3}i}$

$$\text{ix) } \frac{3i^5 + 2i^7 + i^9}{i^6 + 2i^8 + 3i^{18}} \quad \text{x) } \frac{5+7i}{4+3i} + \frac{5+7i}{4-3i}$$

4) Solve the following equations for $x, y \in \mathbb{R}$

i) $(4-5i)x + (2+3i)y = 10-7i$

ii) $(1-3i)x + (2+5i)y = 7+i$

iii) $\frac{x+iy}{2+3i} = 7-i$

iv) $(x+iy)(5+6i) = 2+3i$

v) $2x+i^9y(2+i) = xi^7+10i^{16}$

5) Find the value of

i) $x^3+2x^2-3x+21$, if $x = 1+2i$.

ii) x^3-5x^2+4x+8 , if $x = \frac{10}{3-i}$

iii) $x^3-3x^2+19x-20$, if $x = 1-4i$.

6) Find the square roots of

i) $-16+30i$ ii) $15-8i$ iii) $2+2\sqrt{3}i$

iv) $18i$ v) $3-4i$ vi) $6+8i$

ACTIVITY

Activity 3.1:

Carry out the following activity.

If $x + 2i = 7x - 15i^6y + i^3(y + 4)$, find $x + y$.

Given : $x + 2i = 7x - 15i^6y + i^3(y + 4)$

$$x + 2i = 7x - 15 \square y + \square (y + 4)$$

$$x + 2i = 7x + \square y - (y + 4) \square$$

$$x - \square + 2i = 7x - (y + 4) \square$$

$$\therefore x - 15y - 7x = \square \text{ and } \square = -(y + 4)$$

$$\therefore -6x - 15y = \square \text{ and } y = \square$$

$$\therefore x = \square, y = \square$$

$$\therefore x + y = \square$$

Activity 3.2:

Carry out the following activity

Find the value of $\frac{a+b\omega+c\omega^2}{c+a\omega+b\omega^2}$ in terms of ω^2

$$\begin{aligned} \text{Consider } \frac{a+b\omega+c\omega^2}{c+a\omega+b\omega^2} &= \frac{\square \left(\frac{a}{\omega^2} + \frac{b}{\omega} + c \right)}{c+a\omega+b\omega^2} \\ &= \frac{\left(\frac{a\square}{\omega^3} + \frac{b\square}{\omega^3} + c \right) \omega^2}{c+a\omega+b\omega^2} \\ &= \frac{\left(\frac{a\omega}{1} + \frac{b\square}{\square} + c \right) \omega^2}{c+a\omega+b\omega^2} \\ &= \frac{(\square + \square + c) \omega^2}{c+a\omega+b\omega^2} \\ &= \square \end{aligned}$$

